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Landau Damping and Resonant Energy Absorption*

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A physical interpretation of the Landau damping is given based on the Vlasov equations. It is shown that Landau's discussion and Kidal's calculation of resonant energy absorption are equivalent. It is also found possible to compare the analysis based on the Vlasov equations to Dawson's intuitive formulation.

Author

1. INTRODUCTION

THE problem of plasma oscillations has been discussed by many authors.[1] The well-known solution by Landau who used the Laplace transform method, had shown that in general a damping exists. For a collision-free plasma, this phenomenon seems rather puzzling.

In an attempt to interpret the origin of such damping several theories have been proposed, namely (1) electron trapping,[2] (2) phase mixing,[3] and (3) resonant energy absorption.[4]

In general, the first two theories are difficult to maintain. Some of the argument can be seen from the examples given by Simon.[5] The explanation based on energy absorption by resonant electrons is likely the most reasonable theory. Recently Dawson[6] has given a fairly thorough discussion on this subject. Using the resonant energy absorption model, he is able to show the Landau damping in an intuitive formulation. Furthermore, several points regarding the theory given by Jackson,[2(b)] who considered the electron trapping model, are also clarified in his discussion. Quite independently, Kidal[4(b),7] attempted to explain the Landau damping based on a similar model but started with a different mathematical formulation. He calculated the energy absorption ΔA, i.e.,

$$\Delta A = e\left\langle \int\int\int \mathbf{E}_1\cdot \mathbf{v} f(\mathbf{v}) d^3v \right\rangle_{\mathrm{av}}, \qquad (1)$$

where $\langle\ \rangle$ denotes mean value in time and the subscript "av" denotes average in space. The damping coefficient γ is then determined from the assumption that the time rate of decay of the wave energy must equal the absorbed energy. The general results contained in his two papers are interesting, but there are a few disagreements with previously existing results. For example, the Landau damping in the case of pure longitudinal oscillations obtained by him is different by a factor $\frac{1}{2}$ from that found by Landau.[8] He mentioned that the reason for this deviation may be the different methods of approach. This is not true as is shown later in this note.

The present note makes some remarks on the same subject and may suggest a physically clearer view of Landau's work. In the following, we see that the dispersion relation in Landau's work,[9]

$$\frac{\omega_p^2}{k^2 n_0}\int_C \frac{1}{(u-\omega/K)}\frac{\partial F_0}{\partial u}du - 1 = 0,$$

is equivalent to some energy relation. Our discussion is

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[2] (a) D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851, 1864 (1949); (b) J. D. Jackson, J. Nuclear Energy (Part C) **1**, 171 (1960).

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[4] (a) J. Dawson, Phys. Fluids 4, 869 (1961); (b) A. Kidal, Nuovo cimento **20**, 104 (1961).

[5] A. Simon, Danish Atomic Energy Commission, Riso Rept. 18, 1960 (unpublished), pp. 61–100.

[6] J. Dawson, Phys. Fluids **4**, 869 (1961).

[7] A. Kidal, J. Nuclear Energy (Part C) **3**, 256 (1961).

[8] Actually the difference is by a factor $(-\frac{1}{2})(-1)$, as we shall see later.

[9] Notation is explained in later discussion.

based on the Vlasov equations. However, we see that some of the discussion is comparable with Dawson's theory. It is the author's opinion that if Landau damping is a physically sensible phenomenon, we ought to be able to explain it within the scope of the Vlasov equations. This is the main purpose of the following discussion.

2. ENERGY CONSERVATION AND DISPERSION RELATION

It is not difficult to show that in the absence of external fields, the linearized Vlasov equations can be decoupled into longitudinal and transverse components. In the present discussion, we consider only the longitudinal component since Landau damping does not appear in transverse waves.

The longitudinal component of the linearized Vlasov equations actually contains three equations

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\nabla f_1=-\frac{e}{m}\mathbf{E}_1\cdot\frac{\partial f_0}{\partial \mathbf{v}} \tag{2}$$

$$\nabla\cdot\mathbf{E}_1=4\pi e\int f_1 d^3v. \tag{3}$$

$$\frac{\partial \mathbf{E}_1}{\partial t}=-4\pi e\int \mathbf{v} f_1 d^3v. \tag{4}$$

where f_1 and $\mathbf{E}_1$ are the perturbed distribution function and field, respectively, and f_0 is the equilibrium distribution function. It is important to remark that (3) and (4) are not independent; therefore we can solve the problem of longitudinal oscillation by using Eq. (2) together with either (3) or (4). It is true that (2) and (3) are commonly used.

If we multiply both sides of (4) by $\mathbf{E}_1$, we obtain

$$\frac{\partial}{\partial t}\left(\frac{E_1^2}{8\pi}\right)=-e\int \mathbf{E}_1\cdot\mathbf{v} f_1 d^3v. \tag{5}$$

Clearly, (5) states the conservation of energy corresponding to the first order quantities. The right-hand integral gives the average work done by the electrical field on the electrons or simply the energy transferred to the electrons[10] per unit time per unit volume, and the left side gives the time rate of change of the electrical field energy density. If we take the average in space and in time of Eq. (5), we obtain

$$\left\langle\frac{\partial}{\partial t}\frac{E_1^2}{8\pi}\right\rangle_{\text{av}}=-e\left\langle\int \mathbf{E}_1\cdot\mathbf{v} f_1(\mathbf{v})d^3v\right\rangle_{\text{av}}=-\Delta A, \tag{6}$$

which is precisely the relation used by Kidal who arrived at this relation by intuition.[4(b)]

Since Eq. (3) and Eq. (6) are essentially equivalent, we conclude that Landau's solution must satisfy (6). However, in the following, we give an alternative discussion on Landau damping so that its physical origin is easily displayed.

Let us consider a particular Fourier component of the waves. As we know, if we assume

$$f_1\sim\exp[i(kx-\omega t)],$$
$$E_1\sim\exp[i(kx-\omega t)],$$

where $\mathbf{k}$ is taken to be in the x direction, we can get the same dispersion relation as that based on the Laplace transform method provided the singular integral is properly defined. Then from Eq. (2), f_1 is determined

$$f_1=\frac{ie}{m}\frac{E_1}{k(u-\omega/k)}\frac{\partial f_0}{\partial u}. \tag{7}$$

It can be easily shown that the well-known dispersion relation,

$$1-\frac{\omega_p^2}{k^2 n_0}\int_C\frac{1}{(u-\omega/k)}\frac{\partial F_0}{\partial u}du=0, \tag{8}$$

is obtainable from (5) or (6). In (8), u is the velocity component parallel to $\mathbf{k}$ and

$$F_0(u)=\int f_0(u,\mathbf{v}_\perp)d\mathbf{v}_\perp,$$

where $\mathbf{v}_\perp$ is the vector component of $\mathbf{v}$ normal to $\mathbf{k}$. C denotes the Landau contour.[1(a)] That is, if the root ω/k is located in the upper half of the complex u plane the contour is along the real u axis, and if the root ω/k is in the lower half-plane the contour is first along the real u axis and then around the point $u=\omega/k$ as shown in Fig. 1. However, we are not particularly interested in doing this. Our discussion is mainly concerned with Eq. (6) which is physically more tractable.

3. LANDAU DAMPING AS RESONANT ENERGY ABSORPTION

If we insert Eq. (7) into Eq. (5), we obtain

$$\frac{\partial}{\partial t}\frac{E_1^2}{8\pi}=-\frac{i\omega_p^2}{kn_0}\frac{E_1^2}{4\pi}\int_{-\infty}^{\infty}\frac{u}{u-\omega/k}\frac{\partial F_0}{\partial u}du. \tag{9}$$

Let us be less formal and use Eq. (9) rather than (6). For the long-wavelength limit, we assume *a priori* that $\omega=\omega_r+i\gamma$ and $\gamma/k\rightarrow 0$, $k\rightarrow 0$, and then expand the

[10] We will confine our discussion to the "damped wave" solution only. In general, of course, there may exist "growing wave" solutions. Evidently, the nature of the solutions is dictated by the equilibrium distribution function F_0. It has been shown by F. Berz [Proc. Phys. Soc. (London) **B69**, 939 (1956)] and J. D. Jackson[2(b)] that for an isotropic or single-humped F_0, the growing wave solution cannot exist. All these discussions are beyond the scope of the present paper and therefore are omitted.

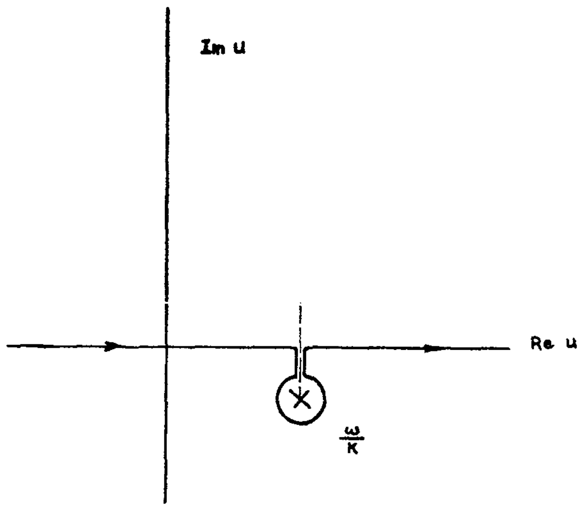

FIG. 1. Landau contour.

integral in Eq. (9) up to $O(\gamma)$. Thus

$$\frac{\partial}{\partial t}\frac{E_1^2}{8\pi}=-\frac{i\omega_p^2}{kn_0}\frac{E_1^2}{4\pi}\int_{-\infty}^{\infty}\frac{u}{u-\omega/k}\frac{\partial F_0}{\partial u}du$$
$$=-\frac{i\omega_p^2}{kn_0}\frac{E_1^2}{4\pi}\left\{P\int_{-\infty}^{\infty}\frac{u}{u-\omega_r/k}\frac{\partial F_0}{\partial u}du+\pi\frac{\gamma\omega_r}{k^2}\frac{\partial^2F_0}{\partial u^2}\right.$$
$$+i\pi\frac{\omega_r}{k}\frac{\partial F_0}{\partial u}\bigg|_{u=\omega_r/K}+i\frac{\gamma}{k}\left[P\int_{-\infty}^{\infty}\frac{1}{u-\omega_r/k}\frac{\partial F_0}{\partial u}du\right.$$
$$\left.\left.+P\int_{-\infty}^{\infty}\frac{u}{u-\omega_r/k}\frac{\partial^2F_0}{\partial u^2}du\right]\right\},\tag{10}$$

where P denotes principal value. For definiteness, we have considered that γ is positive here. However, according to the similar definition of the singular integral given by Landau, it can be easily shown that the present analysis is also valid for negative γ.

Furthermore, it must be remarked here that in order to do this calculation correctly we definitely ought to keep the terms up to $O(\gamma E_1^2/4\pi)$ in the expansion (at least for the imaginary part of the integral). It can be seen that if the last two terms are ignored, we get the incorrect result:

$$\gamma=\frac{\pi\omega_r\omega_p^2}{k^2n_0}\frac{\partial F_0}{\partial u}\bigg|_{u=\omega_r/k}.\tag{11}$$

It is instructive to compare Eq. (10) with Dawson's analysis.[6] For convenience, we also imagine that the plasma may be divided into two parts: the main plasma and the resonant electrons. The main plasma consists of all the electrons with velocities $u\neq\omega_r/k$, and the resonant electrons possess the phase velocity of the wave, $u=\omega_r/k$. Then, it may be visualized that evaluation of the principal part of each of the above integrals is equivalent physically to considering only the main plasma and excluding the resonant electrons.

Furthermore, since the last two terms have coefficients of $O(\gamma)$, we only need to evaluate the two integrals to zeroth order in γ.

We shall make use of the following expressions:

$$P\int_{-\infty}^{\infty}\frac{1}{u-\omega_r/k}\frac{\partial F_0}{\partial u}du=\frac{n_0k^2}{\omega_p^2}+O(\gamma),\tag{12}$$

$$P\int_{-\infty}^{\infty}\frac{u}{u-\omega_r/k}\frac{\partial^2F_0}{\partial u^2}du=\frac{\omega_r}{k}P\int_{-\infty}^{\infty}\frac{1}{u-\omega_r/k}\frac{\partial^2F_0}{\partial u^2}du$$
$$=-\frac{2k^2n_0}{\omega_p^2[1-(k/\omega_r)(d\omega_r/dk)]}+O(\gamma),\tag{13}$$

where (13) is obtainable from (12) by integration by parts. These two terms in Eq. (10) represent part of the contribution from the main plasma.

Since we are only interested in computing the damping coefficient, we need only to consider Eq. (10) in the following form:

$$\frac{2\gamma}{[1-(k/\omega_r)(d\omega_r/dk)]}\left(\frac{E_1^2}{8\pi}\right)=\frac{\pi\omega_p^2\omega_r}{k^2n_0}\left(\frac{E_1^2}{8\pi}\right)\frac{\partial F_0}{\partial u}\bigg|_{u=\omega_r/k},\tag{14}$$

where relations (12) and (13) have been employed.

The right-hand term, which originally came from the δ-function term in the Dirac relation when we expanded the integral in Eq. (9), is the energy absorbed by the resonant electrons. Thus

$$\gamma=\frac{\pi\omega_p^2\omega_r}{2k^2n_0}\frac{\partial F_0}{\partial u}\bigg|_{u=\omega_r/k}\left(1-\frac{k}{\omega_r}\frac{d\omega_r}{dk}\right).\tag{15}$$

For very long wavelengths, the ratio of the group velocity $d\omega_r/dk$ and the phase velocity ω_r/k becomes so small that (15) gives essentially Landau's approximate result[11] if we insert

$$F_0(u)=n_0\left(\frac{m}{2\pi\kappa T}\right)^{\frac{1}{2}}\exp\left(-\frac{mu^2}{2\kappa T}\right).$$

It is interesting to remark that we have the effective longitudinal-wave energy density in the main plasma:

$$U_{\text{wave}}=\frac{E_1^2}{8\pi[1-(k/\omega_r)(d\omega_r/dk)]},$$

which, as well as the result for the resonant absorption,

[11] If we compute γ from (15) carefully based on the Maxwellian distribution, we will find Landau's approximate calculation is bigger by $\exp(\frac{3}{2})$. This error was first pointed out by A. G. Sitenko and K. N. Stepanov, Soviet Phys.—JETP 4, 512 (1957) and later by J. D. Jackson [reference 2(b)].

is in perfect agreement with that calculated by Dawson through an intuitive formulation.

The discussion of the short-wavelength case based on the same model can be done similarly. However, we shall omit it since physically the problem is not interesting.

4. DISCUSSION OF KIDAL'S CALCULATION

As mentioned at the beginning of this paper, the calculation of Landau damping by Kidal gives a result with a factor $\frac{1}{2}$ missing. But in fact the deviation is more than this. In Kidal's discussion[12] the perturbed distribution function f_1 has an incorrect leading sign. If we make this correction, his result for Landau damping would differ from Landau's result by a factor $-\frac{1}{2}$. However, he considers

$$\omega=\omega_r-i\gamma,$$

and integrates the energy absorption integral,

$$\Delta A=-e\int_{-\infty}^{\infty} du\,\langle f(u)Eu\rangle_{\mathrm{av}},$$

along the real axis in the u plane without using the Landau contour. This introduces a second error of sign in front of the term related to the δ function, and therefore eventually the deviation is by a factor $(-\frac{1}{2})(-1)$. The main error which results in the incorrect damping coefficient is due to the fact that during the computation of the resonant energy absorption, i.e.,

$$\Delta A=-\frac{e^2}{2\kappa T}\langle E_0{}^2\exp(-2\gamma t)\rangle \times\int_{-\infty}^{\infty} du\,F_0(u)u^2\frac{1}{\gamma\{1+[(ku-\omega_r)/\gamma]^2\}}, \tag{16}$$

Kidal approximates $(\pi\gamma\{1+[(ku-\omega_r)/\gamma]^2\})^{-1}$ by the δ function $\delta(ku-\omega_r)$. This implies that the integral is only evaluated to zeroth order in γ. As we mentioned before, this is not correct. We must compute the principal part in addition to the δ-function part. With all these remarks the correct expression for ΔA should thus be

$$\Delta A=-\frac{e^2}{2\kappa T}\langle E_0{}^2e^{-2\gamma t}\rangle\left\{P\int_{-\infty}^{\infty} du\,F_0(u)u^2\frac{\gamma}{(ku-\omega_r)^2} -\pi\frac{\omega_r{}^2}{k^3}F_0\left(\frac{\omega_r}{k}\right)\right\}. \tag{17}$$

[12] See, for instance, reference 4(b), Eq. (25).

This first integral is evidently equivalent to the last two terms in Eq. (10) of our previous discussion. (Note: Here F_0 is Maxwellian).

5. CONCLUSIONS

The present note has presented a physical interpretation of Landau's dispersion relation,[13] with the following results:

(1) Landau's dispersion relation and Kidal's calculation of energy absorption have been shown to be essentially equivalent.

(2) A brief comparison with Dawson's analysis has been made and the general results are in complete agreement.

(3) The errors in Kidal's calculation have been pointed out.

Since in the linearized theory it is required that

$$\frac{1}{\gamma}\ll\left(\frac{m}{k|E_0|e}\right)^{\frac{1}{2}}\approx T_t \text{ (trapping time)},$$

the trapping theory, as pointed out by Dawson previously, is not acceptable to explain the Landau damping which is found in the linear theory. The origin of the damping is mainly due to the energy absorption of the resonant electrons. However, one remark which should be made here is that from the result for the damping coefficient we see that, in order to have damping well defined for a finite time interval, we must have a continuous distribution $F_0(u)$ at least in the vicinity of $u=\omega_r/K$. This is certainly reasonable. Since, for a discrete distribution,

$$F_0(u)=\delta(u-\omega_r/k),$$

the resonant electrons will be out of phase with the wave after absorbing energy from the wave and being accelerated during an infinitesimal time interval.

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[13] We have used the name "dispersion relation" loosely since, strictly speaking, Eq. (8) should not be called the dispersion relation.